

Light Monopoles, Electric Vortices and Instantons

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Abstract

A Quantum Field Theory for magnetic monopoles is described and its phase structure fully analysed.

1

The strong coupling limit of $N = 3D2$ supersymmetric gauge theory in four dimensions has been studied in depth by Seiberg and Witten in a brilliant series of papers [1]. An important outcome is that the 'strongly coupled' vacuum turns out to be a weakly coupled theory of light magnetic monopoles. Moreover, a version of electric-magnetic duality fitting the Montonen-Olive conjecture [2] proves to be true in the model.

In this letter we offer a brief description of the physics involved in the Seiberg-Witten action. There are two kinds of quanta: light monopoles and dual-photons. Monopole-monopole as well as dual-photon-monopole interactions are due to the vertices coming from the non-quadratic terms of the Lagrangian. This is the content of perturbation theory although different topological sectors show important non-perturbative effects in the model. We find topological defects as the solutions of a minor modification, a sign, of the monopole equations [3]. The change yields non-trivial regular solutions in \mathbf{R}^4 which, on physical grounds, we find to be either electric vortices or instantons. As a consequence, the phase structure of the theory is determined.

We start from the euclidean action

$$\begin{aligned}
 S = 3D \int d^4x & \left\{ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} (D_\mu \Psi_+)^+ D_\mu \Psi_+ + \frac{1}{4} (D_\mu \Psi_+)^+ \Sigma_{\mu\nu} D_\nu \Psi_+ \right\} \\
 & + \int d^4x \left\{ i \frac{\lambda_1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} j_{\mu\nu} + \frac{\lambda_2^2}{8} j_{\mu\nu} j_{\mu\nu} \right\}.
 \end{aligned} \tag{1}$$

Ψ_+ is a right-handed Weyl spinor, the light monopole field. A_μ is a dual electromagnetic potential. $D_\mu \Psi_+ = 3D \partial_\mu \Psi_+ + ig A_\mu \Psi_+$ is the covariant derivative with respect to the gauge group $U(1)_d$, the dual of the electromagnetic $U(1)$. The magnetic charge, $g = 3D \frac{4\pi}{e}$ is the

coupling constant and the tensor field is $F_{\mu\nu} = 3D\partial_\mu A_\nu - \partial_\nu A_\mu$. Finally, $\Sigma_{\mu\nu} = 3D\frac{1}{2}[\gamma_\mu, \gamma_\nu]$ where, γ_μ are the euclidean γ -matrices in chiral representation: if σ_i are the Pauli matrices,

$$\gamma_\mu = 3Di \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad \begin{matrix} \sigma_\mu & = 3D & (i1_2, \sigma_i) \\ \bar{\sigma}_\mu & = 3D & (i1_2, -\sigma_i) \end{matrix} \quad \gamma_s = 3D \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\Psi(x) = 3D \begin{pmatrix} \Psi_L(x) \\ \Psi_R(x) \end{pmatrix} \quad \Psi_\pm(x) = 3D \frac{1 \pm \gamma_5}{2} \Psi(x)$$

and $j_{\mu\nu}(x) = 3D\frac{1}{2}\Psi_+^\dagger(x)\Sigma_{\mu\nu}\Psi_+(x)$.

The unusual kinetic terms in the spinor fields are possible because in euclidean space there is $SO(4)$ symmetry: there are no linear scalars in the derivatives for Weyl spinors, (see [4]). These are due to the duality of the monopoles with respect to the scalar fields of the parent theory, the $N = 3D2$ supersymmetric Yang-Mills-Higgs theory [5]. Despite the existence of terms of the form $j_{\mu\nu}j_{\mu\nu}$ the theory is renormalizable; the dimension of Ψ_+ is 1, a non-crucial fact because we take the model as physically relevant in the strong e , weak g , limit.

The $j - F$ interaction term in (1) tells us that physically $j_{\mu\nu}$ is due to the electric dipolar momentum of the monopole. We deal with a spinor representation of the Lorentz group twisted by the $U(1)$ group of electro-magnetic duality transformations carrying a property that we call m -spin. The m -spin, or m -helicity in the massless case, yields an intrinsic electric dipolar momentum in the magnetic monopoles and labels the representations of this strange version of the Lorentz group which presumably comes from some twisting of the original $N = 3D2$ SYMH theory.

In this model, on the other hand, all the fields are in the adjoint representation of the $SU(2)$ gauge group. Thus, the spin of the monopole is zero meanwhile the dyonic excitations have an integer spin: the monopole field, despite its spinorial nature, must be quantized according to Bose statistics.

Perturbation theory tells us that there are two kinds of quanta:

1. Light Monopoles. The momentum space propagator is:

$$\Delta_E^M(k) = 3D \frac{k_\mu = (\delta_{\mu\nu} + \bar{\sigma}_{\mu\nu})k_\nu}{k^4} \quad , \quad \begin{matrix} \bar{\sigma}_{ij} & = 3D & i\varepsilon_{ijk}\sigma_k = \\ \bar{\sigma}_{i4} & = 3D & i\sigma_i = 3D - \bar{\sigma}_{4=i} \end{matrix} .$$

2. Dual-photons. The euclidean propagator is:

$$\Delta_{E\mu\nu}^d(k) = 3D \frac{1}{k^2} (\delta_{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2}) .$$

We also read from the non-quadratic terms of the Lagrangian two kinds of vertices:

- a. A quartic monopole self-interaction:

$$V_{MM} = 3D\lambda_2^2 \bar{\sigma}_{\mu\nu} \bar{\sigma}_{\mu\nu}$$

- b. A dual-photon/monopole vertex:

$$V_{MD}(k) = 3Dgk_\mu + \lambda_1 \varepsilon_{\mu\nu\rho\sigma} \bar{\sigma}_{\nu\rho} k_\sigma$$

Physical features: besides renormalizability, at least at the same level as scalar QED, perturbation theory is conformally invariant and the electric dipolar momentum contributes both to the MD and the MM vertices; not only the magnetic charge g and the λ_2 coupling are important.

The amplitude for monopole-monopole scattering at the lowest order in perturbation theory and $\alpha = 3D1$, the Feynman gauge, is:

$$\begin{aligned} T_{MM}(k'_2, k_2; k'_1, k_1) &= 3DU_{1/2}^\dagger(k'_2)(gk_\mu + \lambda_1 \varepsilon_{\mu\nu\rho\sigma} \bar{\sigma}_{\nu\rho} k_\sigma) U_{1/2}(k_2) \\ &\quad \frac{1}{k^2} \cdot U_{1/2}^\dagger(k'_1)(gk_\mu + \lambda_1 \varepsilon_{\mu\alpha\beta\gamma} \bar{\sigma}_{\alpha\beta} k_\gamma) U_{1/2}(k_1) \\ &\quad \delta^{(4)}(k_1 + k_2 - k'_1 - k'_2) + \lambda_2^2 U_{1/2}^+(k'_2) \bar{\sigma}_{\mu\nu} U_{1/2}(k_2) \\ &\quad U_{1/2}^+(k'_1) \bar{\sigma}_{\mu\nu} U_{1/2}(k_1) \delta^{(4)}(k_1 + k_2 - k'_1 - k'_2) \end{aligned}$$

if $k_\mu = 3Dk'_{2\mu} - k_{2\mu} = 3Dk'_{1\mu} - k_{1\mu}$ and $U_{1/2}(k)$ are the plane wave spinors:

$$U_{1/2}(k) = 3D \begin{pmatrix} 1 \\ \frac{k_1 + ik_2}{|k| + k_3} \end{pmatrix}.$$

2

To study the non-perturbative regime, it is convenient to implement a Bogomolny splitting [6]: defining the self-dual part of $F_{\mu\nu}$ as $F_{\mu\nu}^+ = 3D\frac{1}{2}(F_{\mu\nu} + \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma})$ we write:

$$\begin{aligned} S &= 3D \int d^4x \left\{ \frac{1}{2} (F_{\mu\nu}^+ - \frac{i}{2} \lambda_2 j_{\mu\nu})^2 + \frac{1}{4} (D_\mu \Psi_+)^+ = \gamma_\mu \gamma_\nu D_\nu \Psi_+ \right\} \\ &+ \int d^4x \left\{ \frac{i\lambda_2}{2} F_{\mu\nu}^+ j_{\mu\nu} + \frac{1}{2} (D_\mu \Psi_+)^+ \Sigma_{\mu\nu} D_\nu \Psi_+ + \frac{i\lambda_1}{4} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \Psi_+^\dagger = \Sigma_{\mu\nu} \Psi_+ \right\} \\ &- \int d^4x \{ \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \} \end{aligned} \quad (2)$$

If $\lambda_2 = 3Dg = 3D - \lambda_1$, a critical point analogous to that occurring between type I and type II superconductors, (2) becomes

$$\begin{aligned} S &= 3D \int d^4x \left\{ \frac{1}{2} (F_{\mu\nu}^+ - \frac{i}{2} g j_{\mu\nu})^2 + \frac{1}{4} (D_\mu \Psi_+)^+ \gamma_\mu \gamma_\nu D_\nu \Psi_+ \right\} \\ &- \int d^4x \{ \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \} \end{aligned}$$

and re-scaling $x \rightarrow \frac{1}{g}x$ we see that at the critical point S is almost $\frac{1}{g^2}$ times the Seiberg-Witten action plus a topological term looking similar to the Pontryagin or second Chern number. There is a different relative sign in the first squared term; we choose the sign of the Pauli momentum interaction, the term $j - F$ term, in (1) by assigning the negative m -spin projection to the monopole quanta. This amounts to a different choice of orientation in \mathbf{R}^4 with respect to the convention assumed in the Seiberg-Witten theory due to the specific choice of the twisting in the $N = 3D2$ SYMH parent theory.

Solutions of the first order system,

$$F_{\mu\nu}^+ = 3D\frac{1}{2}\Psi_+^\dagger \Sigma_{\mu\nu} \Psi_+ = \quad (3.a)$$

$$\gamma_\mu D_\mu \Psi_+ = 3D0 \quad (3.b)$$

are absolute minima of the euclidean action and therefore play an important rôle in the system. In order to find them, it is convenient to split (3.a-b) into components: (3.a) reads

$$F_{12}^+ = 3D - \frac{1}{2}(\phi_1^* \phi_1 - \phi_2^* \phi_2) = 3DF_{34}^+ \quad (4.a)$$

$$F_{23}^+ = 3D - \frac{1}{2}(\phi_1^* \phi_2 + \phi_2^* \phi_1) = 3DF_{14}^+ \quad (4.b)$$

$$F_{13}^+ = 3D - \frac{i}{2}(\phi_1^* \phi_2 - \phi_2^* \phi_1) = 3DF_{24}^+ \quad (4.c)$$

where $\Psi_R = 3D \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. Also, we have for (3.b)

$$(D_3 + iD_4)\phi_1 + i(D_1 - iD_2)\phi_2 = 3D0 \quad (5.a)$$

$$(D_1 + iD_2)\phi_1 - i(D_3 - iD_4)\phi_2 = 3D0 \quad (5.b)$$

Multiplying (5.a) by $(D_1 + iD_2)$, (5.b) by $(D_3 + iD_4)$, subtracting, using (4.b-c) and integrating over all \mathbf{R}^4 we obtain

$$\int d^4x \{ |\phi_1|^2 |\phi_2|^2 + |D_{z_1} \phi_2|^2 + |D_{z_2} \phi_2|^2 \} = 3D0 \quad (6)$$

$D_{z_1} = 3DD_1 - iD_2$, $D_{z_2} = 3DD_3 - iD_4$, after a partial integration. We are then left with two possibilities:

- A. $\phi_2 = 3D0$. Solutions with m -spin 1/2 in the x_3 -direction $\frac{1}{2}\sigma_3 \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} = 3D\frac{1}{2} \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$
- B. $\phi_1 = 3D0$ and $D_{z_1} \phi_2 + D_{z_2} \phi_2 = 3D0$. Solutions with m -spin projection $-1/2$:

$$\frac{1}{2}\sigma_3 \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} = 3D - \frac{1}{2} \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix}.$$

Formula (6) tells us that type A and B exclude each other: even for solitons, only one of the two polarizations is possible.

3

Searching for explicit solutions we make the following ansatz adapted to type A:

$$\phi_1(x) = 3D\phi_1(x_1, x_2) \quad \phi_2 = 3DA_3 = 3DA_4 = 3D0$$

$$A_1(x) = 3DA_1(x_1, x_2); \quad A_2(x) = 3DA_2(x_1, x_2)$$

The only non-zero (4)-(5) equations reduce to

$$\begin{aligned} F_{12}(x_1, x_2) &= 3D - |\phi_1(x_1, x_2)|^2 \\ &= 7F(D_1 + iD_2)\phi_1(x_1, x_2) = 3D0 \end{aligned} \quad (7)$$

It is well known that system (7) is tantamount to the Liouville equation in \mathbf{R}^2 , [7], and the general solution such that $\lim_{r_1 \rightarrow \infty} \phi(x_1, x_2) = 3D0$, where $r_1^2 = 3Dx_1^2 + x_2^2$, guaranteeing finite energy density is, see [8],

$$\phi_1^{[k]}(z_1) = 3D \frac{2f'(z_1)V^2(z_1)}{|V(z_1)|^2 + |f(z_1)V(z_1)|^2} \quad z_1 = 3Dx_1 + ix_2 \quad (8)$$

$$f(z) = 3Df_0 + \sum_{i=3D1}^k \frac{c_i}{z - z^{(i)}} = V(z) = 3D \prod_{(i=3D1)}^k (z - z^{(i)})$$

The solutions $\phi_1^{[k]}$ have infinite action,

$$S[\phi_1^{[k]}] = 3D \frac{2\pi k}{g^2} \lim_{L, T \rightarrow \infty} \int_{-\frac{L}{2}}^{-\frac{L}{2}} dx_3 \int_{\frac{T}{2}}^{-\frac{T}{2}} = dx_4$$

although the electric flux is finite: $\Phi_E[\phi_1^{[k]}] = 3D \int d^2x F_{12} = 3D \frac{2\pi}{g} k$.

Observe that k is positive and the flux is located around $z_1^{(i)}$, the zeroes of $\phi_1(z_1)$. It spreads out, however, with $|c_i|$, the length scale of the solution, which is a free parameter due to the scale invariance of the theory. There is also freedom in choosing $\arg c_i$ because the $U(1)_d$ symmetry and the moduli space of solutions is \mathbf{C}^{2k} : the parameters are the centers of the solitons $z_1^{(i)}$ and the modulus and phase of c_i , determining the scale and phase of each individual soliton.

Solutions of type B are given by the complementary ansatz:

$$\begin{aligned} \phi_1 = 3DA_1 = 3DA_2 = 3D0 \quad , \quad \phi_2(x) = 3D\phi_2(x_3, x_4 =) \\ A_3(x) = 3DA_3(x_3, x_4) \quad ; \quad A_4(x) = 3DA_4(x_3, x_4) \end{aligned}$$

We now meet the system

$$\begin{aligned} F_{34}(x_3, x_4) &= 3D|\phi_2(x_3, x_4)|^2 \\ (D_3 - iD_4)\phi_2(x_3, x_4) &= 3D0 \end{aligned} \tag{9}$$

again leading to the Liouville equations with the right sign to obtain non-singular finite energy density solutions:

$$\phi_2^{[k]}(\bar{z}_2) = 3D \frac{2f'(\bar{z}_2)V^2(\bar{z}_2)}{|V(\bar{z}_2)|^2 + |f(\bar{z}_2)V^2(\bar{z}_2)|^2}, \quad \bar{z}_2 = 3Dx_3 - ix_4. \tag{10}$$

The euclidean action is

$$S[\phi_2^{[k]}] = 3D \frac{2\pi = k}{g^2} \lim_{L \rightarrow \infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx_1 \int_{-\frac{L}{2}}^{\frac{L}{2}} dx_2,$$

there is no need to change the sign of euclidean time, but the 'euclidean' magnetic flux

$$\Phi_M[\phi_2^{[k]}] = 3D \int \int dx_3 dx_4 F_{34} = 3D - \frac{2\pi}{g} k =$$

is negative, fitting the sign of the m -spin projection. The moduli space of these solutions is also \mathbf{C}^{2k} .

We can understand the topological origin of these solutions in the following way: consider \mathbf{R}^4 as $\mathbf{R}^2 \otimes \mathbf{R}^2$. The finite energy density conditions, alternatively in each \mathbf{R}^2 according to the type of solutions, plus the conformal invariance of the equations make the problem tantamount to solving the system on $S^2 \times S^2$ [10]. We have two topological numbers: $n_{12} = 3Dk$, electric flux, and $n_{34} = 3D - k$, euclidean magnetic flux, labelling the topological

sectors of the theory. In fact we have more: setting the x_4 -coordinate to be euclidean time there are two= other possibilities for splitting \mathbf{R}^4 as $\mathbf{R}^2 \otimes \mathbf{R}^2$. Choosing a basis in which σ_1 is diagonal and the Kähler form is $\omega = 3Ddx_2 \wedge dx_3 + dx_4 \wedge dx_1$, similar ansatzes yield

$$\begin{aligned} F_{23}(x_2, x_3) &= 3D - |\phi_1(x_2, x_3)|^2 & (D_2 + iD_3)\phi_1(x_2, x_3) &= 3D0 & : \text{ type A} \\ F_{14}(x_2, x_4) &= 3D|\phi_2(x_1, x_4)|^2 & (D_1 - iD_4)\phi_2(x_1, x_4) &= 3D0 & : \text{ type B} \end{aligned}$$

or σ_2 diagonal, $\omega_2 = 3Ddx_3 \wedge dx_1 + dx_2 \wedge dx_4$,

$$\begin{aligned} F_{13}(x_1, x_3) &= 3D - |\phi_1(x_1, x_3)|^2 & (D_1 + iD_3)\phi_1(x_1, x_3) &= 3D0 & : \text{ type A} \\ F_{24}(x_2, x_4) &= 3D|\phi_2(x_2, x_4)|^2 & (D_2 - iD_4)\phi_2(x_2, x_4) &= 3D0 & : \text{ type B} \end{aligned}$$

There are electrically charged solitons in 3 different planes, labelled by,

$$e_i = 3D\varepsilon_{ijk}n_{jk} \quad , \quad n_{jk} = 3D - n_{kj} \quad , \quad e_i \in \mathbf{Z}^+$$

and also magnetically charged solitons, labelled by,

$$m_i = 3Dn_{i4} \quad , \quad m_i \in \mathbf{Z}^-$$

although the a 'priori' conserved topological numbers m_i will not survive the Wick rotation to real time.

We can extend the theory by modifying the spinor tensor/spinor tensor interaction in the following way:

$$j_{\mu\nu}^{v_a}(x) = 3D\frac{1}{2}\Psi_+^\dagger(x)\Sigma_{\mu\nu}\Psi_+(x) - \frac{1}{2}v_a^2n_a^\dagger\Sigma_{\mu\nu}n_a,$$

no summation in a , with $n_1 = 3D\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $n_2 = 3D\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ = and $a = 3D1, 2$. With a

re-scale of variables $\Psi_+ \rightarrow v_a\Psi_+$, $A_\mu \rightarrow v_aA_\mu$ and $x_\mu \rightarrow \frac{1}{gv_a}x_\mu$, the ansatz of type A plus $v_2 = 3D0$, produces the system:

$$\begin{aligned} F_{12} &= 3D1 - |\phi_1|^2 \\ (D_1 + iD_2)\phi_1 &= 3D0 = \end{aligned} \tag{11}$$

Solutions of (11) such that $\lim_{r_1 \rightarrow \infty} \phi_1(x_1, x_2) = 3D1$ exist and have been thoroughly analysed in [9]. They are electric vortices, flux tubes, with the flux concentrated around the zeroes of $\phi_1(x)$: there is neither freedom of scale (because the theory is not conformally invariant) nor phase freedom (because $U(1)_d$ symmetry is broken spontaneously). The moduli space is \mathbf{C}^k and the same $e_i = 3D\varepsilon_{ijk}n_{jk}$ topological quantum numbers are conserved, $e_i \in \mathbf{Z}^+$.

Alternatively, the ansatz of type B and $v_1 = 3D0$ leads to the system:

$$\begin{aligned} F_{34} &= 3D|\phi_2(x_3, x_4)|^2 - 1 \\ (D_3 - iD_4)\phi_2(x_3, x_4) &= 3D0 \end{aligned} \tag{12}$$

and there are 'euclidean' magnetic anti-vortices of negative integer flux concentrated around the zeroes of ϕ_2 . The moduli space is \mathbf{C}^k and the topological quantum numbers $m_i = 3Dn_{4i} \in \mathbf{Z}^-$ will not be conserved because of the euclidean time character of x_4 .

4

In order to study the phase structure of the model, it is convenient to have a richer set of solutions. Adding a left handed Weyl spinor Ψ_- , the anti-monopole field with m -spin projection $-1/2$, a system of equations like (3.a)- (3.b) with F^+ replaced by F^- arises. There are solutions of the kind (8) and (10) with electric and magnetic fluxes of opposite sign.

Similar solutions to those described above have been studied by the author in Reference [11]. In that case, the Hopf invariant forced flux tube pairs. Also 't Hooft electric and magnetic flux lines found in pure Yang-Mills in a box [12] share many physical features with the Seiberg-Witten solutions. We study the quantum behaviour of the last type of solitons in 't Hooft's framework.

In the $A_4 = 3D0$ gauge, consider the operator

$$\hat{B}(C, t) = 3D \exp \left[ig \left(\int d^3x [\hat{\vec{E}}(\vec{x}, t) \int_C = ds \vec{A}^A(\vec{y}(s)) + \hat{\Pi}_1^*(\vec{x}, t) \int_C ds \phi_1^A(\vec{y}(s))] \right) \right]$$

$$\hat{\Pi}_1(\vec{x}, t) = 3D - i(\dot{\hat{\phi}}_1(\vec{x}, t) + \frac{1}{2} D_{z_1} \hat{\phi}_2(\vec{x}, t) + D_3 \hat{\phi}_1(\vec{x}, t)),$$

which creates an electric flux smeared by ϕ_1 along the curve C : a type A solution for which the zero of the ϕ_1 field with multiplicity e_3 is repeated along C . Choosing C as the x_3 -axis, perhaps with periodic boundary conditions $x_3(-L_3) = 3Dx_3(L_3)$, $\hat{B}(C, t)$ creates an electric solitonic string of electric flux $n_{12} = 3De_3$ which is conserved for topological reasons. On coherent states of the Hilbert state; the action of $\hat{B}(C, t)$ is

$$\hat{B}(C, t) |\vec{A}, \vec{E}; \phi_1, \phi_2\rangle_0 \propto |\vec{A} + L_3 A^A(x_1, x_2), \vec{E}; \phi_1 + L_3 \phi_1^A(x_1, x_2), \phi_2\rangle_{e_3} =$$

The proportionality factor is due to the fact that $\hat{B}(C, t)$ measures magnetic flux.

Quantum states related to type B solutions are more difficult to analyse because they correspond to a tunnel effect between different vacua when the euclidean time component x_4 is Wick-rotated to real time. In fact there are non-homotopically trivial gauge transformations of the form:

$$\Omega_{k_3}(\vec{x}) = 3D e^{\frac{i2\pi g a_3}{L_3} x_3} \Omega(x_1, x_2), \quad g a_3 = 3D k_3$$

These act on the quantum states by means of an unitary operator:

$$\hat{\Omega}_{k_3}(\vec{x}) |\vec{A}, \vec{E}; \phi_1, \phi_2\rangle_{e_3} = 3D e^{i\omega(k_3)} |\vec{A}, \vec{E}; \phi_1, \phi_2\rangle_{e_3}$$

The group law $\hat{\Omega}_{k_3} \hat{\Omega}_{k'_3} = 3D \hat{\Omega}_{k_3+k'_3}$, $\omega(k_3) + \omega(k'_3) = 3D \omega(k_3 + k'_3)$ and the requirement of the same action for all the gauge homotopy classes $\omega(k_3) = 3D \omega(k_3 + n)$ yields: $\omega(k_3) = 3D 2\pi \theta_3 k_3$. To understand the physical origin of this angle, consider the 'smeared' Wilson operator:

$$\hat{A}(C, t) = 3D \exp \left[ig \left(\int d^3x [\hat{\vec{A}}(\vec{x}, t) \int_C = ds \vec{E}^B(\vec{y}(s)) + \hat{\Pi}_2^*(\vec{x}, t) \int_C ds \phi_2^B(\vec{y}(s))] \right) \right]$$

$$\hat{\Pi}_2(\vec{x}, t) = 3D - i(\dot{\hat{\phi}}_2(\vec{x}, t) + \frac{1}{2} D_{z_1} \hat{\phi}_1(\vec{x}, t) + D_3 \hat{\phi}_2(\vec{x}, t)),$$

$\hat{A}(C, t)$ creates a solution of type B, in the A_4 -gauge, along C . For T long enough and $C \equiv x_3$ -axis,

$$\begin{aligned} \hat{A}(C, t)|\vec{A}, \vec{E}; \phi_1, \phi_2\rangle_{e_3} &\propto |\vec{A} + \Omega(\vec{x}, T), \vec{E}; \phi_1 \Omega(\vec{x}, T), \phi_2\rangle_{e_3} \\ \lim_{t \rightarrow -\frac{T}{2}} \Omega(\vec{x}, t) &= 3D\Omega_0(\vec{x}), \quad \lim_{t \rightarrow \frac{T}{2}} \Omega(\vec{x}, t) = 3D\Omega_{k_3}(\vec{x}), \quad = k_3 = 3Dm_3 \end{aligned}$$

$\hat{A}(C, t)$ creates magnetic flux but the tunnel effect, and the associated instanton angle, means that only strips with flux parametrized by θ_3 are conserved.

Due to the other possible choices of splitting \mathbf{R}^4 in $\mathbf{R}^2 \times \mathbf{R}^2$ there are quantum states characterized by $(\vec{e}, \vec{\theta})$, 3 integers and 3 angles, corresponding to the solutions of the Seiberg-Witten solutions. This analysis is a semi-classical one but sufficient because it is the regime within which we trust the theory. The free energy of each state is thus known and the phases are as follows:

1. $v_a^2 = 3D0$. Both electric and magnetic flux spread out over all the space and we are in the Coulomb phase.
2. $v_1^2 = 3D0$. In this phase there is a constant background $\vec{\theta}$ magnetic field in the $\vec{\theta}$ -vacua ground states. In the *SPA*,

$$\langle \theta_3 | e^{-\hat{H}T} | \theta_3 \rangle \propto \exp\{e^{-2\pi L_1 L_2 / g^2} K L_3 T \cos \theta_3\}$$

where K is a determinantal factor accounting for the gauge and spinor fluctuations in the presence of an instanton up to the one-loop order.

3. $v_2^2 = 3D0$. There is electric order: electric vortices of integral magnetic charge exist and we are in a phase of electric charge confinement. Note that due to the peculiar m -spin properties of the solitons only massless particles are confined.
4. Finally, if $v_a^2 \leq 0$, there are no solutions of the first order equations. This is the normal phase.

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